Transactions Briefs

Control of the Chaotic Duffing Equation with Uncertainty in All Parameters

Antonio Loria, Elena Panteley, and Henk Nijmeijer

***Abstract*—In this brief, we deal with the open problem of controlling the periodically forced Duffing equation with uncertainty in all parameters. To date, several control schemes have been proposed to adapt for the *linearly* appearing unknown parameters but no solution exists for the case when the frequency of the periodic forcing is also unknown. We prove for the *state* feedback control case, global asymptotic convergence for constant and time-varying references. We extend these results to the *position* feedback case and prove global ultimate boundedness.**

I. Introduction

Motivated by potential applications in physics, engineering, and communication theory, various control of complex (chaotic) nonlinear systems has received an increasing interest [1], [4], [9]. Notably, the Oh–Grebogi–York (OGY) method, introduced in [8], focuses on the stable tracking of an unstable periodic orbit in the chaotic attractor of the nominally uncontrolled dynamics. Several possible methods, depending on the desired behavior of the system, have been developed (see, for example, the review in [4]), but a complete analysis of the resulting closed-loop system has been given only in a few cases. In particular, satisfactory state feedback control results for the controlled forced Duffing equation are given, for instance, in [2], [3], [6]. Since the periodically forced Duffing equation exhibits chaotic motion for suitable parameter settings, this system forms an important illustration for controlling a chaotic system. Chronologically, a tracking state feedback controller was established in [2] and extended to an output feedback tracking controller in [6], which also deals with the case of parameter uncertainties. See also [3], where a "speed gradient" adaptive controller was proposed. In both works, the authors deal only with uncertainty in those parameters which appear *linearly*. In this brief, we prove for the state feedback control case, set-point convergence, respectively, tracking for the controlled Duffing equation, with uncertainty in *all* parameters. If only position feedback can be used, we prove that the closed-loop error dynamics converge to a (arbitrarily small) neighborhood of the origin, thus extending the results of [3] and [6].

Manuscript received February 6, 1997; revised April 6, 1998. This work was supported in part by the Systems and Control group of the University of Twente and in part by the European Capital and Human Mobility project under Grant ERB 4050PL930138. This paper was recommended by Associate Editor U. Helmke.

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Publisher Item Identifier S 1057-7122(98)08050-7.

Problem Formulation: We consider the controlled periodically forced Duffing equation

$$\ddot{x} + p_1\dot{x} + p_2 x + p_3 x^3 = q\cos\omega t + u \tag{1}$$

where $u \in \mathbb{R}$ is the control input. Assume that the parameters p_i, $i = 1, 2, 3$, q, and ω are unknown but some constants p_M and q_M are known such that $p_M \geq \max_i\{|p_i|\}$ and $q_M \geq q$. Under these conditions, for (1) with *any* initial conditions define a feedback control u such that $\lim_{t\to\infty}\tilde{x}(t) = 0$ where $\tilde{x}(t) \triangleq (x(t) - x_d(t))$ and $x_d(t)$ is any twice continuously differentiable desired trajectory.

II. Main Results

In this section, we give our main results. Our first two results deal with state feedback solutions to the set-point stabilization, respectively tracking control problem for (1). In order to cope with the uncertainty on parameters p_i, $i = 1, 2, 3$, we propose a PID-based controller to solve the set-point problem and an adaptive controller for the tracking problem.

Proposition 2.1 (PID Set-Point Control): Consider system (1) with a set-point $x_d = \text{const}$ and the PID-based control input

$$\begin{aligned} u =& -k_1\tilde{x} - k_d\dot{x} + \nu - k_2\tilde{x}^3 - k_3(x^3 - x_d^3) \\ & - q_M\,\mathrm{sgn}(\dot{x} + \varepsilon\tilde{x}) \end{aligned} \tag{2}$$

$$\dot{\nu} = -k_i\tilde{x} \tag{3}$$

where k_1, k_2, k_3, k_d, and k_i are positive constants such that $k_3 > p_M$ and $k_1 > 4p_M(\varepsilon + 1) + 1$, with $\varepsilon > k_i > 0$. Then there exists a sufficiently small k_i such that the closed-loop system (1)–(3) is globally asymptotically convergent, that is, for *any* initial conditions we have that $|\nu(t)| \in \mathcal{L}_\infty$ and $\lim_{t\to\infty} z(t) \triangleq \mathrm{col}[\tilde{x}(t), \dot{x}(t)] = \mathbf{0}$.□

Proposition 2.2 (Adaptive Tracking Control): Consider system (1) and the control law

$$\begin{aligned} u =& \ddot{x}_d - k_1\tilde{x} - k_2\tilde{x}^3 - k_d\dot{\tilde{x}} + \hat{p}_1\dot{x}_d + \hat{p}_2 x_d \\ & + \hat{p}_3(x_d^3 + 3x_d\tilde{x}x) - q_M\,\mathrm{sgn}(\dot{\tilde{x}} + \varepsilon\tilde{x}) \end{aligned} \tag{4}$$

$$\dot{\hat{\theta}} = -\gamma(\dot{\tilde{x}} + \varepsilon\tilde{x})\phi(x, x_d, \dot{x}_d) \tag{5}$$

where γ is a small positive constant, $\min\{k_1, k_2, k_d\} > p_M + \varepsilon^2$ and we have defined

$$\hat{\theta} = \mathrm{col}[\hat{p}_1, \hat{p}_2, \hat{p}_3] \tag{6}$$

$$\phi(x, x_d, \dot{x}_d) = \mathrm{col}[\dot{x}_d, x_d, x_d^3 + 3x_d\tilde{x}x]. \tag{7}$$

Then the closed-loop system (1), (4) is globally asymptotically convergent, uniformly in t. □

Notice that the controller of Proposition 2.1 is built on a PID structure: the first three terms in the control law, two correspond exactly to those of a PID controller. The terms $-k_2\tilde{x}^3 - k_3(x^3 - x_d^3)$ are added in order to dominate the nonlinearity x^3, and finally, the term $-q_M\,\mathrm{sgn}(\dot{x} + \varepsilon\tilde{x})$ is used to dominate the *bounded* disturbance $q\cos\omega t$. The addition of this nonsmooth feedback term is the novelty of our approach with respect to previous results which assume that ω is known, see, for instance, [3], [5]. Our third result ensures that the trajectories of the closed-loop system converge to a bounded domain which may be made arbitrarily small by enlarging the control gains.

Proposition 2.3 (PI²D Set-Point Control): Consider system (1) and assume that p_3 is *known*. The PI²D-based controller

$$u = -k_1\tilde{x} - k_d\vartheta + \nu - k_2\tilde{x}^3 - k_3(x^3 - x_d^3) - q_M\,\mathrm{sgn}(\tilde{x} - \vartheta) \tag{8}$$

$$\dot{\nu} = -k_i(\tilde{x} - \vartheta) \tag{9}$$

$$\dot{x}_c = -a(x_c + bx + \xi) \tag{10}$$

$$\vartheta = x_c + bx + \xi \tag{11}$$

$$\dot{\xi} = -\frac{\varepsilon b}{k_d}[(k_3 + p_3)(x^3 - x_d^3) + k_2\tilde{x}^3 + k_1'\tilde{x}] \tag{12}$$

where k_1, k_2, k_3, k_d, a, b, and k_i are positive constants such that $k_3 > p_M$, $k_1 > 4p_M(\varepsilon + 1) + 1$, with $\varepsilon > k_i > 0$ sufficiently small; this guarantees that the closed loop system (1), (8)–(12) is globally ultimately bounded (GUB), that is, for *any* initial conditions $\lim_{t\to\infty}\|z(t)\| \triangleq \lim_{t\to\infty}\|[\tilde{x}(t), \dot{x}(t)]\| \le \eta$ where $\eta > 0$ is such that $\lim_{\substack{b\to\infty \\ k_1'\to\infty}} \eta = 0$. □

It should also be pointed out that the PI²D-based controller of Proposition 2.3 is based upon the PI²D controller for robot manipulators, originally proposed in [7]. As in the standard PID case, the first three terms of control law defined in (8) correspond exactly to the control law used in [7]. The rest of the terms are added to dominate the nonlinearity x^3 and the perturbation $q\cos\omega t$. Since in this case no velocity measurements are considered, we have added the nonlinear dynamic extension (10)–(12).

A. Proof of the Main Results

Proof of Proposition 2.1: First, in order to simplify the notation, let us define $\zeta = \nu - p_2x_d - p_3x_d^3 - (k_i/\varepsilon)\tilde{x}$ and partition $k_1 = k_1' + (k_i/\varepsilon)$ where we impose $0 < k_i < \varepsilon$ and $k_1' > 4p_M(\varepsilon+1)$. Then we can write the closed-loop equations (1), (2) in the more convenient form

$$\ddot{x} + (k_d + p_1)\dot{x} + (k_1' + p_2)\tilde{x} + k_2\tilde{x}^3 + (k_3 + p_3)(x^3 - x_d^3) = \zeta - q_M\,\mathrm{sgn}(\dot{x} + \varepsilon\tilde{x}) + q\cos\omega t \tag{13}$$

$$\dot{\zeta} = -k_i\tilde{x} - \frac{k_i}{\varepsilon}\dot{x}. \tag{14}$$

Now consider the Lyapunov function candidate $V(\tilde{x}, \dot{x}, \zeta) = V_1(\tilde{x}) + V_2(\tilde{x}, \dot{x}, \zeta)$ where

$$V_1(\tilde{x}) = \tfrac{1}{4}\left\{k_2\tilde{x}^4 + (k_3 + p_3)[x^4 - x_d^4 - 4\tilde{x}x_d^3]\right\} + \tfrac{1}{2}\left[k_1' + p_2 + \varepsilon(p_1 + k_d)\right]\tilde{x}^2 \tag{15}$$

$$V_2(\tilde{x}, \dot{x}, \zeta) = \frac{\varepsilon}{2k_i}\zeta^2 + \tfrac{1}{2}\dot{x}^2 + \varepsilon\tilde{x}\dot{x}. \tag{16}$$

By defining $V_1'(\tilde{x}) \triangleq V_1(\tilde{x}) - \frac{1}{4}k_1'\tilde{x}^2$ and using the definition of k_1, it is not difficult to verify that $V_1(\tilde{x})$ is positive definite and radially unbounded provided that the conditions of Proposition 2.1 hold. Besides, one can verify that $V_2'(\tilde{x}, \dot{x}, \zeta) \triangleq V_2(\tilde{x}, \dot{x}, \zeta) + \frac{1}{4}k_1'\tilde{x}^2$ is positive definite and radially unbounded if $k_1' > 2\varepsilon^2$, which is satisfied for sufficiently small ε. Finally, we conclude that $V(\tilde{x}, \dot{x}, \zeta) = V_1'(\tilde{x}) + V_2'(\tilde{x}, \dot{x}, \zeta)$ is positive definite and radially unbounded. Next we proceed to evaluate the time derivative of V along the trajectories of the closed-loop system (13), (14), that is, $\dot{V} = \dot{V}_1 + \dot{V}_2$. After some straightforward bounding, we obtain

$$\dot{V} \le -\varepsilon(k_3 + p_3)(x^3 - x_d^3)\tilde{x} - (k_d + p_1 + \varepsilon)\dot{x}^2 - \varepsilon(k_1' + p_2)\tilde{x}^2 - \varepsilon k_2\tilde{x}^4 - (q_M\,\mathrm{sgn}(\dot{x} + \varepsilon\tilde{x}) - q\cos\omega t)(\dot{x} + \varepsilon\tilde{x}). \tag{17}$$

Now, notice that the first term on the right-hand side of (17) is nonpositive since $\mathrm{sgn}(x^3 - x_d^3) = \mathrm{sgn}(\tilde{x})$. Moreover, notice that since $|q\cos\omega t| \le |q| < q_M$ for all $t \in \mathbb{R}$, the last term of (17) is bounded by $-(q_M\,\mathrm{sgn}(\dot{x} + \varepsilon\tilde{x}) - q\cos\omega t)(\dot{x} + \varepsilon\tilde{x}) \le -(q_M - |q|)|\dot{x} + \varepsilon\tilde{x}|$, thus $\dot{V}(\dot{x}, \tilde{x}, \zeta)$ is negative semi-definite

$$\dot{V} \le -(k_d + p_1 + \varepsilon)\dot{x}^2 - \varepsilon(k_1' + p_2)\tilde{x}^2 \le 0. \tag{18}$$

The proof is completed noticing that (18) implies that $V(z(t), \zeta(t))$ is a decreasing function of time, hence V is bounded and consequently $\dot{x}, \tilde{x}, \zeta \in \mathcal{L}_\infty$. Moreover, from (13) we obtain that also $\ddot{x} \in \mathcal{L}_\infty$. Inequality (18) also implies that $\tilde{x}, \dot{x} \in \mathcal{L}_2$. Then we conclude that both $\tilde{x}(t) \to 0$ and $\dot{x}(t) \to 0$ as $t \to \infty$ [10]. ■

Proof of Proposition 2.2: The proof essentially follows along the lines of the proof of Proposition 2.1.

Proof of Proposition 2.3: Taking into account the change of variable $\zeta = \nu - p_2x_d - p_3x_d^3 - (k_i/\varepsilon)\tilde{x}$ and the partition of $k_1 = k_1' + (k_i/\varepsilon)$, we write the closed-loop equation (1), (8)–(12) in the more convenient form

$$\ddot{x} + p_1\dot{x} + (k_1' + p_2)\tilde{x} + k_2\tilde{x}^3 + (k_3 + p_3)(x^3 - x_d^3) - k_d\vartheta = \zeta + q\cos\omega t - q_M\,\mathrm{sgn}(\tilde{x} - \vartheta) \tag{19}$$

$$\dot{\zeta} = -k_i(\tilde{x} - \vartheta) - \frac{k_i}{\varepsilon}\dot{x} \tag{20}$$

$$\dot{\vartheta} = -a\vartheta + b\dot{x} + \dot{\xi} \tag{21}$$

where $\dot{\xi}$ is defined in (12). Notice that $\dot{\xi}$ is a function of the position error only, hence the complete state of the closed-loop system (19)–(21) is $\mathrm{col}[\tilde{x}, \dot{x}, \vartheta, \zeta]$. Now consider the function $V(\tilde{x}, \dot{x}, \zeta, \vartheta) = V_4(\tilde{x}) + V_5(\tilde{x}, \dot{x}, \zeta, \vartheta)$ where we have defined for convenience

$$V_4(\tilde{x}) = \frac{1}{4}\left(1 - \frac{\varepsilon^2 b}{k_d}\right) \cdot [(k_3 + p_3)(x^4 - x_d^4 - 4\tilde{x}x_d^3) + k_2\tilde{x}^4 + k_1'\tilde{x}^2] + \frac{1}{4}(k_1' + 2p_2 + 2\varepsilon p_1)\tilde{x}^2$$

$$V_5(\tilde{x}, \dot{x}, \zeta, \vartheta) = \frac{\varepsilon}{2k_i}\zeta^2 + \frac{1}{2}\dot{x}^2 + \varepsilon(\tilde{x} - \vartheta)\dot{x} + \frac{k_d}{2b}\vartheta^2.$$

After lengthy straightforward calculations omitted here for lack of space, one can prove that $V(\tilde{x}, \dot{x}, \zeta, \vartheta)$ is positive definite with a global and unique minimum at the origin if $k_3 > p_M$ and

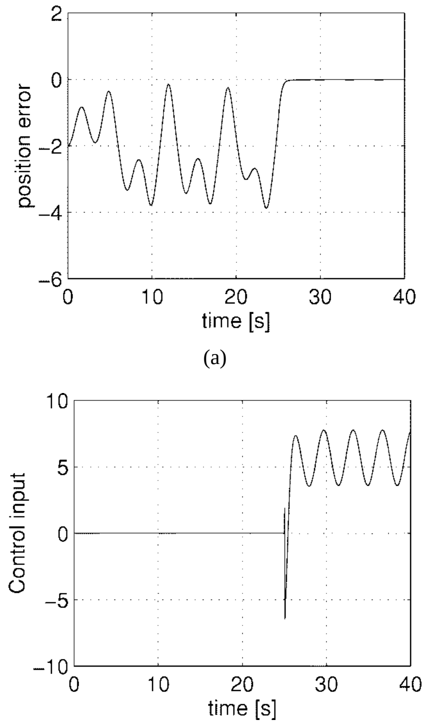
$$\frac{k_d}{b} > 2\varepsilon^2 \tag{22}$$

which holds under the conditions of Proposition 2.3. Also, V is radially unbounded if $k_1' > 8\varepsilon^2$, which holds for sufficiently small ε. Using inequality $-[\dot{x} + \varepsilon(\tilde{x} - \vartheta)](q_M\,\mathrm{sgn}(\tilde{x} - \vartheta) - q\cos\omega t) \le 2q_M|\dot{x}| - \varepsilon(q_M - |q\cos\omega t|)|\tilde{x} - \vartheta| \le 2q_M|\dot{x}|$, after some straightforward bounding we obtain that the time derivative of V along the trajectories of (19)–(21) is

$$\begin{aligned}\dot{V} \le &-\frac{1}{2}\underbrace{[\varepsilon b - 2\varepsilon - (\varepsilon + 2)p_M - \varepsilon a]}_{2\alpha_1}\dot{x}^2 \\ &- \frac{1}{2}\underbrace{\left[2\frac{k_d a}{b} - \varepsilon(3k_d + a + 2p_M)\right]}_{2\alpha_2}\vartheta^2 - \frac{\varepsilon}{2}\underbrace{[2k_1' + 3p_M + k_d]}_{2\alpha_3}\tilde{x}^2 \\ &- \varepsilon k_2\tilde{x}^4 - \varepsilon(k_3 + p_M)(x^3 - x_d^3)\tilde{x} - \frac{\varepsilon b}{2}\dot{x}^2 + 2q_M|\dot{x}|\end{aligned} \tag{23}$$

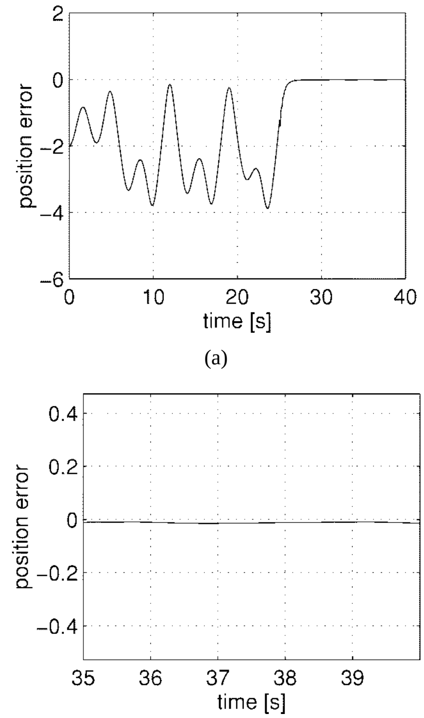
where the addition of the last two terms is negative if $\dot{x} \notin B_{\eta_1}$ where

$$B_{\eta_1} \triangleq \left\{\dot{x} \in \mathbb{R}: |\dot{x}| \le \eta_1,\ \eta_1 = \frac{4q_M}{\varepsilon b}\right\}. \tag{24}$$



(b)

Fig. 1. Duffing's error dynamics and control input with the PID-based controller. (a) Position error. (b) Control input.



(b)

Fig. 2. Duffing's error dynamics and control input with the PI^{2}D-based controller. (a) Position error. (b) Position input.

Thus, for all $\dot{x} \notin B_{\eta_1}$ and $\mathrm{col}[\tilde{x}, \zeta, \vartheta] \in \mathbb{R}^3$, we have as before that (23) implies that $V(z(t), \zeta(t), \vartheta(t))$ is a decreasing function of time, hence bounded and consequently $\dot{x}, \tilde{x}, \zeta \in \mathcal{L}_\infty$. Moreover, from (19) we obtain also that $\ddot{x} \in \mathcal{L}_\infty$. Inequality (23) also implies that $\tilde{x}, \dot{x} \in \mathcal{L}_2$. Then we conclude that both $\tilde{x}(t) \to 0$ and $\dot{x}(t) \to 0$ as $t \to t_{\eta_1}$ where t_{η_1} is the time instant when $|\dot{x}(t_{\eta_1})| \in B_{\eta_1}$. Next we analyze the closed-loop trajectories for all time $t \geq t_{\eta_1}$. Notice from (23) that for all $\dot{x} \in B_{\eta_1}$ and $\mathrm{col}[\tilde{x}, \zeta, \vartheta] \in \mathbb{R}^3$ we have that

$$\dot{V} \leq -\frac{\varepsilon b}{2}\eta_1^2 + 2q_M\eta_1 - \varepsilon\alpha_3\tilde{x}^2$$

from which we deduce that $\dot{V} \leq -(\varepsilon\alpha_3/2)\tilde{x}^2$ for all $\tilde{x} \notin B_{\eta_2}$ where we defined

$$B_{\eta_2} \triangleq \left\{\tilde{x} \in \mathbb{R}: |\tilde{x}| \leq \eta_2, \quad \eta_2 = \sqrt{\frac{\varepsilon b\eta_1^2 + 4q_M\eta_1}{\varepsilon\alpha_3}}\right\}. \tag{25}$$

Thus, using standard arguments, we conclude that $\lim_{t\to\infty} \tilde{x}(t) \leq \eta_2$ and $\lim_{t\to\infty} \dot{x}(t) \leq \eta_1$. Finally, we prove that in the limit case when $b, k_1' = \infty$, then $\eta_1, \eta_2 = 0$. From the definition of η_1 in (24), it is clear that $\eta_1 \to 0$ as $b \to \infty$. Now, substituting the value of η_1 from (24) in the definition of η_2, that is, (25), we obtain

$$\eta_2 = \left(\frac{16q_M^2}{\varepsilon^2 b(2k_1' + 3p_M + k_d)}\right)^{1/2}. \tag{26}$$

It is easy to see that by increasing b $\eta_2 \to 0$ for any fixed ε, k_1', and k_d. Also for any fixed k_d we can make b arbitrarily large while ε arbitrarily small so that (22) is satisfied. Hence defining without loss of generality $k_d \triangleq \beta\varepsilon^2 b$, with $\beta > 2$ so that we can rewrite (26) as

$$\eta_2 = \left(\frac{16q_M^2\beta}{k_d(2k_1' + 3p_M + k_d)}\right)^{1/2}$$

and notice that $\eta_2 \to 0$ as $k_1' \to \infty$ for any ε and b. Since there is no contradiction in the conditions, the GUB thesis is proved. ■

III. Simulation Results

On SIMULINK of MATLAB, the Duffing equation parameters were selected as $p_1 = 0.4, p_2 = -1.1, p_3 = 1, q = 2.1$, and $\omega = 1.8$, in which case the system has a chaotic behavior [2]. To compare, we set the gains for both the PID- and PI^{2}D-based controllers to $k_1 = 40$, $k_2 = 30$, $k_3 = 30$, $k_d = 100$, $k_i = 0.2$, and $\varepsilon = 0.21$ and desired reference $x_d = 2$. We have approximated $\mathrm{sgn}(\xi) \cong \tanh(\mu\xi)$ with $\mu = 100\,000$. The controller is applied for $t \geq 25$ s. In Fig. 1, we show that after a short transient the control goal $\lim_{t\to\infty} \tilde{x}(t) \to 0$ is achieved by means of the state feedback PID-based controller of Proposition 2.1 while the control input converges to an oscillating signal with frequency and amplitude ω and q, respectively, which compensates for the perturbing signal $q \cos \omega t$. Notice that, for this particular case, the control input is relatively small for all time.

In Fig. 2 we show the error dynamics of the Duffing equation driven by the PI^{2}D-based controller of Proposition 2.3 with $a = 10$ and $b = 8$. Even though it is hard to appreciate in the figure, $\|\tilde{x}(t)\| < 0.015$ for all $t \geq 35$. However, as proved in the last section, one may decrease this bound by enlarging k_1, the price paid being a large control input. To compare, we have computed the integral square errors (ISE's) $\int_0^{40} \tilde{x}^2(s)\,dt$ for both set-point controllers. For the PID-based controller, we obtained ISE $= 148.7235$ while for the PI^{2}D-based controller ISE $= 148.4175$. Fig. 2 depicts also the control input for the PI^{2}D controller. Notice that, in this case, even though all PID gains were selected as for the previous controller, the control input is considerably large. The explanation to this is that for the PI^{2}D controller we use in the control input the term $k_d\vartheta = k_d(x_c + bx + \xi)$, roughly speaking the control input grows at rate $k_d b = 800$. Hence, control gain b affects considerably the magnitude of the control effort.

In the third simulation, we show the performance of the controller of Proposition 2.2 with $k_1 = 60$, $k_2 = 10$, $k_3 = 10$, $k_d = 60$, and $\gamma = 0.1$ and desired reference $x_d(t) = \sin(t)$. As for the two previous simulations, the control action was applied for $t \geq 25$. Fig.

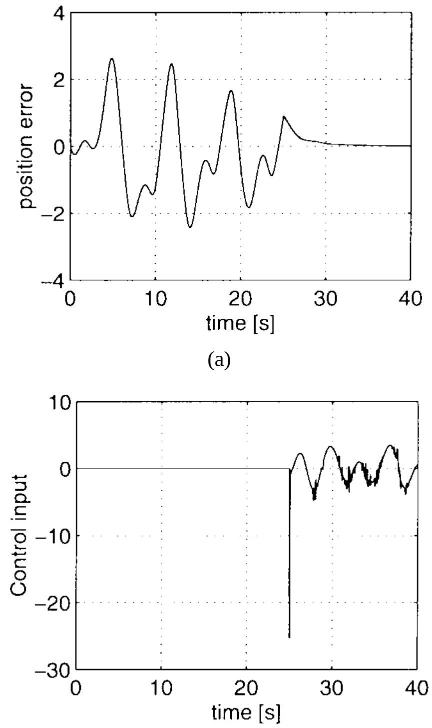


Fig. 3. Duffing's error dynamics under adaptive tracking controller of Proposition 2.2. (a) Position error. (b) Control input.

3 shows the error dynamics of the Duffing equation and the control input.

IV. Concluding Remarks

We have addressed the open problem of stabilization of the Duffing equation with uncertainty in *all* parameters. First, we have solved the set-point control problem by using a PID-based controller. Second, we proposed an adaptive tracking controller which guarantees global asymptotic convergence. Our third result applies to the case unmeasurable velocities. We have proposed a PI^{2}D-based controller which guarantees that the position error converges to an arbitrarily small bounded domain. We have shown in simulations that our theoretical results match perfectly with what was expected.

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Chaos Due to the Interaction of High- and Low-Frequency Modes

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***Abstract*—The results of theoretical and experimental investigations of the interaction of high- and low-frequency oscillators in the presence of a periodic perturbation are discussed. The reasons for threshold of chaos to be low in such a system are clarified in terms of the development of the decay instability and the destruction of quasiperiodic oscillations. Analytical criteria for predicting the onset of chaos are provided.**

***Index Terms*—Chaos, coupled mode analysis, nonlinear systems, varactors.**

I. Introduction

It is well known that the interaction of two or more modes (oscillators) may result in the appearance of chaotic oscillations. The particular conditions of the chaos onset strongly depend on individual properties of the interacting modes such as the degree and type of their nonlinearity, their frequency ratio, type of coupling, etc. (see, [1]–[5]). By now, the most extensively studied systems are those with the resonant interaction of modes, when the following resonant condition is met:

$$\omega_1/\omega_2 \approx n/m, \tag{1}$$

where ω_1 and ω_2 are natural frequencies of the interacting modes, n and m are comparatively small integers. Along with this, various parametric resonances of the form:

$$\omega \approx \omega_1 \pm \omega_2, \tag{2}$$

where ω stands for the frequency of an external force or the third interacting mode, are typical for autonomous or nonautonomous multimode systems. The recent results described in [6] have shown that even the interaction of two modes with substantially different natural frequencies, i.e., when

$$\omega_1 \gg \omega_2 \tag{3}$$

Manuscript received April 4, 1997; revised November 25, 1997 and February 5, 1998. This work was supported in part by European Community (COPERNICUS Program) under Contract ERBIC15CT960816 and by the International association for the promotion of cooperation with scientists from the New Independent States of the former Soviet Union grant under Grant INTAS-UA 95-139. This paper was recommended by Associate Editor J. A. Nossek.

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Publisher Item Identifier S 1057-7122(98)08379-2.